JOURNAL OF COMPUTATIONAL PHYSICS 119, 244–251 (1995)

Strategies for the Accurate Computation of Potential Derivatives in Boundary Element Method: Application to Two-Dimensional Problems

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Received October 5, 1993; revised November 15, 1994

This paper describes two strategies for the accurate computations of potential derivatives in boundary element methods. The first method regularizes the quasi singularity in a fundamental solution by referring the potential and its derivatives at the boundary point nearest to a calculation point in a domain. In the second method, a system of coupled equations for an unknown potential and its derivatives at a calculation point is solved to improve accuracy. Green's theorem unifies the derivation of the above methods, which are shown to be suitable for computer implementation. Numerical results show that the present methods considerably improve the accuracy in the computations of potential derivatives. The errors in the present methods are analyzed to evaluate their performance for general cases. Although this paper describes the regularization methods for only two-dimensional problems, it is suggested that those can be easily extended to three-dimensional problems.

1. INTRODUCTION

In computations of a potential and its derivatives in a domain, as well as in those of the elements in system matrices, the boundary element method (BEM) usually employs the Gaussian quadrature for the boundary integrals including a fundamental solution, potentials, and their derivatives (e.g., 1). The accuracy in those computations, however, becomes extremely poor as a calculation point approaches the boundaries, due to the quasi singularity of the fundamental solution. For this reason, several methods for improving the accuracy in BEM have been proposed so far. Although some regularization techniques, e.g., based on subtraction of Taylor polynomials, for singular integrals have been well known in potential theory, we restrict our reference to the methods relevant to the regularization of the quasi singularities in BEM below.

For the computation of potentials, Kisu *et al.* [2] have regularized the quasi singularity in the normal derivative of a fundamental solution using the potential difference between a calculation point and the point on the boundary nearest to the calculation point (boundary reference point). Enokizono *et al.* [3, 4] have evaluated the numerical error in boundary integrals and introduced a formula for ameliorating the accuracy of them. Sladek *et al.* [5] have obtained a non-singular boundary integral representation of the first-order derivatives of potentials using the boundary reference points. Their method, in which the strong singularity in the derivatives of a fundamental solution is relaxed by partial integration, seems complicated for the extension of it for the higher derivatives. The method in Refs. [3, 4] has been extended by Koizumi *et al.* [6] to that for the higher derivatives. Although they obtained good accuracy by their method, they had to choose appropriate harmonic functions in their formula and an arbitrary constant remained. Hayami *et al.* [7] have provided a coordinate transform technique for the evaluation of singular and quasi singular integrals. The method, however, requires a number of integration points for highly accurate computations.

In this paper, we introduce new formulas for the accurate computation of potential derivatives, which include the methods in Refs. [2, 3, 4] in the lowest order. The formulas are readily derived from Green's theorem without arbitrariness and have simple forms suitable for computer implementation.

The remainder of this paper will be organized as follows. In the next section, the two methods are derived by applying Green's theorem to the combination of an algebraic function with a fundamental solution. The third section provides the numerical results for two simple potential problems to assess the reliability of the present methods. In the fourth section, the numerical errors in the present methods, as well as those in the conventional method, are evaluated for general cases. The last section includes some concluding remarks.

2. FORMULAS FOR POTENTIAL DERIVATIVES

We here restrict our consideration to two-dimensional potential problems. The present methods can be easily extended to those for three-dimensional or axisymmetric problems. In addition, we consider only the zeroth, first, and second order derivatives of a potential, which are thought to be of importance for the application in physics and engineering sciences. Figure

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1 illustrates the inner and outer potential problems under consideration. In Fig. 1, x_i $(i = 1, 2)$ are the Cartesian coordinates with the basis vectors $\mathbf{x}_i$, $\mathbf{n}$ and $\mathbf{t}$ denote the normal and tangential unit vectors on the boundaries, respectively, and Γ_∞ represents the boundary at infinity. The vectors $\mathbf{n}$ and $\mathbf{t}$ are in appropriate directions for the inner and outer problems as shown in Fig. 1. Moreover, the potential and its derivatives are assumed to approach zero sufficiently fast as $\mathbf{r} \to \infty$ such that the line integrals on Γ_∞ vanish.

In conventional BEM, the formulas for a potential $u(\mathbf{r})$ and its derivatives at a calculation point $\mathbf{r} = (x_1, x_2) \in \Omega$

$$u(\mathbf{r}) = \int_\Gamma \left[\frac{\partial u(\mathbf{r}')}{\partial n'} G(\mathbf{r}'; \mathbf{r}) - u(\mathbf{r}') \frac{\partial G(\mathbf{r}'; \mathbf{r})}{\partial n'} \right] d\Gamma', \tag{1}$$

$$\frac{\partial u(\mathbf{r})}{\partial x_i} = \int_\Gamma \left[\frac{\partial u(\mathbf{r}')}{\partial n'} \frac{\partial G(\mathbf{r}'; \mathbf{r})}{\partial x_i} - u(\mathbf{r}') \frac{\partial^2 G(\mathbf{r}'; \mathbf{r})}{\partial x_i \partial n'} \right] d\Gamma', \tag{2}$$

$$\frac{\partial^2 u(\mathbf{r})}{\partial x_j \partial x_i} = \int_\Gamma \left[\frac{\partial u(\mathbf{r}')}{\partial n'} \frac{\partial^2 G(\mathbf{r}'; \mathbf{r})}{\partial x_j \partial x_i} - u(\mathbf{r}') \frac{\partial^3 G(\mathbf{r}'; \mathbf{r})}{\partial x_j \partial x_i \partial n'} \right] d\Gamma', \tag{3}$$

are derived through Green's theorem

$$\int_\Omega (v \nabla^2 w - w \nabla^2 v)\, d\Omega = \int_\Gamma \left(v \frac{\partial w}{\partial n} - w \frac{\partial v}{\partial n} \right) d\Gamma, \tag{4}$$

where $i, j = 1, 2$ and $G(\mathbf{r}'; \mathbf{r})$ denotes the fundamental solution which satisfies $\nabla^2 G(\mathbf{r}'; \mathbf{r}) + \delta(\mathbf{r}' - \mathbf{r}) = 0$. In (1)–(3), the fundamental solution and its derivatives are expressed in the form

$$G(\mathbf{r}'; \mathbf{r}) = -\frac{1}{2\pi} \log |\mathbf{R}|, \tag{5}$$

$$\nabla G(\mathbf{r}'; \mathbf{r}) = \frac{1}{2\pi} \frac{\mathbf{R}}{|\mathbf{R}|^2},$$

$$\frac{\partial G(\mathbf{r}'; \mathbf{r})}{\partial n'} = -\mathbf{n}' \cdot \nabla G(\mathbf{r}'; \mathbf{r}), \tag{6}$$

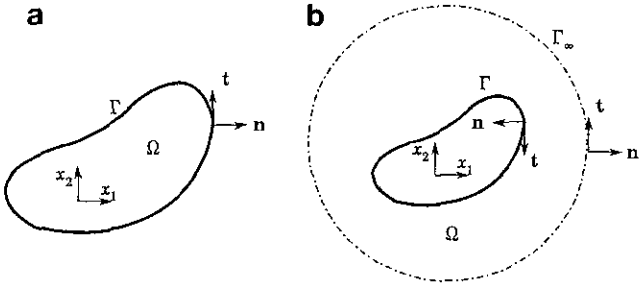

FIG. 1. Two-dimensional potential problems. Ω denotes the domain under consideration: (a) inner problem; (b) outer problem.

$$\nabla\nabla G(\mathbf{r}'; \mathbf{r}) = -\frac{1}{2\pi} \frac{1}{|\mathbf{R}|^4} (|\mathbf{R}|^2 I - 2\mathbf{RR}),$$

$$\nabla \frac{\partial G(\mathbf{r}'; \mathbf{r})}{\partial n'} = -\mathbf{n}' \cdot \nabla\nabla G(\mathbf{r}'; \mathbf{r}), \tag{7}$$

$$\nabla\nabla \frac{\partial G(\mathbf{r}'; \mathbf{r})}{\partial n'} = \frac{1}{\pi |\mathbf{R}|^4} \left[\mathbf{Rn}' + \mathbf{n}'\mathbf{R} + \mathbf{n}' \cdot \mathbf{R} \left(I - 4 \frac{\mathbf{RR}}{|\mathbf{R}|^2} \right) \right], \tag{8}$$

where $\mathbf{R} \equiv \mathbf{r}' - \mathbf{r}$ and I denotes the unit tensor.

When a calculation point $\mathbf{r} \in \Omega$ is in the vicinity of the boundary Γ, the Gaussian quadrature applied to (1)–(3) fails to accurately compute the boundary integrals including the derivatives of the fundamental solution. We modify the formulas (1)–(3) to overcome this problem below.

2.1. *Method A*

We here introduce algebraic functions $u_k(\mathbf{r}'; \mathbf{r}^*)$, $k = 0, 1, 2$, of the points $\mathbf{r}'$ and $\mathbf{r}^* \in \Omega \cup \Gamma$,

$$\begin{aligned} u_0(\mathbf{r}'; \mathbf{r}^*) &= u(\mathbf{r}^*), \\ u_1(\mathbf{r}'; \mathbf{r}^*) &= u_0(\mathbf{r}'; \mathbf{r}^*) + (x_i' - x_i^*) \frac{\partial u(\mathbf{r}^*)}{\partial x_i}, \\ u_2(\mathbf{r}'; \mathbf{r}^*) &= u_1(\mathbf{r}'; \mathbf{r}^*) + \frac{1}{2}(x_i' - x_i^*)(x_j' - x_j^*) \frac{\partial^2 u(\mathbf{r}^*)}{\partial x_j \partial x_i}. \end{aligned} \tag{9}$$

Of course, (9) corresponds to the kth order Taylor expansion of a potential function u around $\mathbf{r}^*$. The normal derivatives of $u_k(\mathbf{r}'; \mathbf{r}^*)$ on Γ can be easily found to be

$$\begin{aligned} \frac{\partial u_0(\mathbf{r}'; \mathbf{r}^*)}{\partial n'} &= 0, \\ \frac{\partial u_1(\mathbf{r}'; \mathbf{r}^*)}{\partial n'} &= n_i' \frac{\partial u(\mathbf{r}^*)}{\partial x_i}, \\ \frac{\partial u_2(\mathbf{r}'; \mathbf{r}^*)}{\partial n'} &= \frac{\partial u_1(\mathbf{r}'; \mathbf{r}^*)}{\partial n'} + n_i'(x_j' - x_j^*) \frac{\partial^2 u(\mathbf{r}^*)}{\partial x_j \partial x_i}, \end{aligned} \tag{10}$$

where $n_i' \equiv \mathbf{n}' \cdot \mathbf{x}_i$.

Introducing now the correspondence

$$(u(\mathbf{r}') - u_0(\mathbf{r}'; \mathbf{r}_0), G(\mathbf{r}'; \mathbf{r})) \to (v, w) \tag{11}$$

in (4), we find a formula for a potential $u(\mathbf{r})$,

$$u(\mathbf{r}) = \int_\Gamma \left[G(\mathbf{r}'; \mathbf{r}) \frac{\partial u(\mathbf{r}')}{\partial n'} - \{u(\mathbf{r}') - u_0(\mathbf{r}'; \mathbf{r}_0)\} \frac{\partial G(\mathbf{r}'; \mathbf{r})}{\partial n'} \right] d\Gamma' + (1 - \eta)\, u(\mathbf{r}_0), \tag{12}$$

where $\mathbf{r}_0$ denotes the boundary reference point nearest to a

calculation point $\mathbf{r} \in \Omega$, and η is a constant which takes its values in the set $\{0, 1\}$ for inner and outer problems, respectively. The constant η here comes from the identity for the outer problem

$$u_k(\mathbf{r}; \mathbf{r}_0) = \int_{\Gamma_\infty} \left[G(\mathbf{r}'; \mathbf{r}) \frac{\partial u_k(\mathbf{r}'; \mathbf{r}_0)}{\partial n'} - u_k(\mathbf{r}'; \mathbf{r}_0) \frac{\partial G(\mathbf{r}'; \mathbf{r})}{\partial n'} \right] d\Gamma'. \tag{13}$$

The formula (12) for the inner problem was introduced in Ref. [2]. One can clearly see in (12) that the quasi singularity in $\partial G(\mathbf{r}'; \mathbf{r})/\partial n'$ is regularized by the difference $u(\mathbf{r}') - u_0(\mathbf{r}'; \mathbf{r}_0)$ which varies as $|\mathbf{r}' - \mathbf{r}_0|$.

Moreover, applying Green's theorem (4) to the combinations

$$\left(u(\mathbf{r}') - u_1(\mathbf{r}'; \mathbf{r}_0), \frac{\partial G(\mathbf{r}'; \mathbf{r})}{\partial x_i} \right) \to (v, w), \tag{14}$$

$$\left(u(\mathbf{r}') - u_2(\mathbf{r}'; \mathbf{r}_0), \frac{\partial^2 G(\mathbf{r}'; \mathbf{r})}{\partial x_j \partial x_i} \right) \to (v, w), \tag{15}$$

and using the identities

$$\int_\Omega g(\mathbf{r}') \frac{\partial \delta(\mathbf{r}' - \mathbf{r})}{\partial x_i} d\Omega' = \left. \frac{\partial g(\mathbf{r}')}{\partial x_i'} \right|_{\mathbf{r}'=\mathbf{r}}, \tag{16}$$

$$\int_\Omega g(\mathbf{r}') \frac{\partial^2 \delta(\mathbf{r}' - \mathbf{r})}{\partial x_j \partial x_i} d\Omega' = \left. \frac{\partial^2 g(\mathbf{r}')}{\partial x_j' \partial x_i'} \right|_{\mathbf{r}'=\mathbf{r}}, \tag{17}$$

and the derivatives of (13) with respect to x_i, we have regularized formulas for the potential derivatives

$$\frac{\partial u(\mathbf{r})}{\partial x_i} = \int_\Gamma \left[\frac{\partial G(\mathbf{r}'; \mathbf{r})}{\partial x_i} \left\{ \frac{\partial u(\mathbf{r}')}{\partial n'} - \frac{\partial u_1(\mathbf{r}'; \mathbf{r}_0)}{\partial n'} \right\} - \{u(\mathbf{r}') - u_1(\mathbf{r}'; \mathbf{r}_0)\} \frac{\partial^2 G(\mathbf{r}'; \mathbf{r})}{\partial x_i \partial n'} \right] d\Gamma' + (1 - \eta) \frac{\partial u(\mathbf{r}_0)}{\partial x_i}, \tag{18}$$

$$\frac{\partial^2 u(\mathbf{r})}{\partial x_j \partial x_i} = \int_\Gamma \left[\frac{\partial^2 G(\mathbf{r}'; \mathbf{r})}{\partial x_j \partial x_i} \left\{ \frac{\partial u(\mathbf{r}')}{\partial n'} - \frac{\partial u_2(\mathbf{r}'; \mathbf{r}_0)}{\partial n'} \right\} - \{u(\mathbf{r}') - u_2(\mathbf{r}'; \mathbf{r}_0)\} \frac{\partial^3 G(\mathbf{r}'; \mathbf{r})}{\partial x_j \partial x_i \partial n'} \right] d\Gamma' + (1 - \eta) \frac{\partial^2 u(\mathbf{r}_0)}{\partial x_j \partial x_i}. \tag{19}$$

One can see again in (18) and (19) that the quasi singularity in the derivatives of the fundamental solution is regularized by the differences $\partial u(\mathbf{r}')/\partial n' - \partial u_k(\mathbf{r}'; \mathbf{r}_0)/\partial n'$ and $u(\mathbf{r}') - u_k(\mathbf{r}'; \mathbf{r}_0)$. In the computations of (18) and (19), the values of the derivatives $\partial u(\mathbf{r}_0)/\partial x_i$ and $\partial^2 u(\mathbf{r}_0)/\partial x_j \partial x_i$ are obtained from the known variables u and $q \equiv \partial u/\partial n$ on Γ as

$$\frac{\partial u(\mathbf{r}_0)}{\partial x_i} = \frac{\partial u(\mathbf{r}_0)}{\partial t} t_i(\mathbf{r}_0) + q(\mathbf{r}_0) n_i(\mathbf{r}_0), \tag{20}$$

$$\frac{\partial^2 u(\mathbf{r}_0)}{\partial x_1^2} = -\frac{\partial^2 u(\mathbf{r}_0)}{\partial x_2^2} = \alpha_1 t_1(\mathbf{r}_0) - \alpha_2 t_2(\mathbf{r}_0), \tag{21}$$

$$\frac{\partial^2 u(\mathbf{r}_0)}{\partial x_1 \partial x_2} = \alpha_1 t_2(\mathbf{r}_0) + \alpha_2 t_1(\mathbf{r}_0), \tag{22}$$

$$\alpha_i = \frac{\partial^2 u(\mathbf{r}_0)}{\partial t^2} t_i(\mathbf{r}_0) + \frac{\partial u(\mathbf{r}_0)}{\partial t} \frac{dt_i(\mathbf{r}_0)}{dt} + \frac{\partial q(\mathbf{r}_0)}{\partial t} n_i(\mathbf{r}_0) + q(\mathbf{r}_0) \frac{dn_i(\mathbf{r}_0)}{dt}, \tag{23}$$

where $t_i \equiv \mathbf{t} \cdot \mathbf{x}_i$. A technique for finding the boundary reference point is described in Ref. [5] in detail.

The method derived in this section is referred to as Method A in the remainder of this paper.

2.2. *Method B*

By replacing the boundary reference point $\mathbf{r}_0$ for a calculation point $\mathbf{r}$ in the correspondence (11), we obtain an alternative formula for a potential as

$$0 = \int_\Gamma \left[G(\mathbf{r}'; \mathbf{r}) \frac{\partial u(\mathbf{r}')}{\partial n'} - \{u(\mathbf{r}') - u_0(\mathbf{r}'; \mathbf{r})\} \frac{\partial G(\mathbf{r}'; \mathbf{r})}{\partial n'} \right] d\Gamma' - \eta u(\mathbf{r}). \tag{24}$$

At this time, the difference $u(\mathbf{r}') - u_0(\mathbf{r}'; \mathbf{r})$ which varies as $|\mathbf{r}' - \mathbf{r}|$ relaxes the quasi singularity in $\partial G(\mathbf{r}'; \mathbf{r})/\partial n'$. After a little rearrangement, (24) becomes

$$u(\mathbf{r}) = \int_\Gamma \left[G(\mathbf{r}'; \mathbf{r}) \frac{\partial u(\mathbf{r}')}{\partial n'} - u(\mathbf{r}') \frac{\partial G(\mathbf{r}'; \mathbf{r})}{\partial n'} \right] d\Gamma' \Bigg/ \left[\eta - \int_\Gamma \frac{\partial G(\mathbf{r}'; \mathbf{r})}{\partial n'} d\Gamma' \right], \tag{25}$$

from which one can compute the value of a potential. The formula (25) for the inner problem was found by evaluating the numerical error in boundary integrals in Refs. [3, 4]. Note here that (25) is closely related to (12) through the selection of $\mathbf{r}^*$ in u_k.

On the other hand, since the function $u_i(\mathbf{r}'; \mathbf{r})$ includes the three unknown quantities, i.e., the potential and derivatives of the first order at a calculation point in itself, application of Green's theorem to $u_i(\mathbf{r}'; \mathbf{r})$ leads to a system of coupled equations for the unknowns, in contrast to the formula obtained

in the previous section. We must, therefore, now consider the combinations

$$\begin{aligned}&(u(\mathbf{r}) - u_1(\mathbf{r}';\mathbf{r}), G(\mathbf{r}';\mathbf{r})) \to (v, w),\\ &\left(u(\mathbf{r}) - u_1(\mathbf{r}';\mathbf{r}), \frac{\partial G(\mathbf{r}';\mathbf{r})}{\partial x_i}\right) \to (v, w)\end{aligned} \tag{26}$$

in (4) to get the linear equations for the unknowns $u(\mathbf{r})$ and $\partial u(\mathbf{r})/\partial x_i$,

$$\begin{aligned}0 = \int_\Gamma \Bigg[& G(\mathbf{r}';\mathbf{r}) \left\{\frac{\partial u(\mathbf{r}')}{\partial n'} - \frac{\partial u_1(\mathbf{r}';\mathbf{r})}{\partial n'}\right\}\\ & - \{u(\mathbf{r}') - u_1(\mathbf{r}';\mathbf{r})\}\frac{\partial G(\mathbf{r}';\mathbf{r})}{\partial n'}\Bigg] d\Gamma' - \eta u(\mathbf{r}),\\ 0 = \int_\Gamma \Bigg[& \frac{\partial G(\mathbf{r}';\mathbf{r})}{\partial x_i}\left\{\frac{\partial u(\mathbf{r}')}{\partial n'} - \frac{\partial u_1(\mathbf{r}';\mathbf{r})}{\partial n'}\right\}\\ & - \{u(\mathbf{r}') - u_1(\mathbf{r}';\mathbf{r})\}\frac{\partial^2 G(\mathbf{r}';\mathbf{r})}{\partial x_i \partial n'}\Bigg] d\Gamma' - \eta\frac{\partial u(\mathbf{r})}{\partial x_i}.\end{aligned} \tag{27}$$

Similarly, the combinations

$$\begin{aligned}&(u(\mathbf{r}') - u_2(\mathbf{r}';\mathbf{r}), G(\mathbf{r}';\mathbf{r})) \to (v, w),\\ &\left(u(\mathbf{r}') - u_2(\mathbf{r}';\mathbf{r}), \frac{\partial G(\mathbf{r}';\mathbf{r})}{\partial x_i}\right) \to (v, w),\\ &\left(u(\mathbf{r}') - u_2(\mathbf{r}';\mathbf{r}), \frac{\partial^2 G(\mathbf{r}';\mathbf{r})}{\partial x_j \partial x_i}\right) \to (v, w)\end{aligned} \tag{28}$$

applied to (4) yield the linear equations for the second-order derivatives with a potential and the first-order derivatives as follows:

$$\begin{aligned}0 = \int_\Gamma \Bigg[& G(\mathbf{r}';\mathbf{r}) \left\{\frac{\partial u(\mathbf{r}')}{\partial n'} - \frac{\partial u_2(\mathbf{r}';\mathbf{r})}{\partial n'}\right\}\\ & - \{u(\mathbf{r}') - u_2(\mathbf{r}';\mathbf{r})\}\frac{\partial G(\mathbf{r}';\mathbf{r})}{\partial n'}\Bigg] d\Gamma' - \eta u(\mathbf{r}),\\ 0 = \int_\Gamma \Bigg[& \frac{\partial G(\mathbf{r}';\mathbf{r})}{\partial x_i}\left\{\frac{\partial u(\mathbf{r}')}{\partial n'} - \frac{\partial u_2(\mathbf{r}';\mathbf{r})}{\partial n'}\right\}\\ & - \{u(\mathbf{r}') - u_2(\mathbf{r}';\mathbf{r})\}\frac{\partial^2 G(\mathbf{r}';\mathbf{r})}{\partial x_i \partial n'}\Bigg] d\Gamma' - \eta\frac{\partial u(\mathbf{r})}{\partial x_i},\\ 0 = \int_\Gamma \Bigg[& \frac{\partial^2 G(\mathbf{r}';\mathbf{r})}{\partial x_j \partial x_i}\left\{\frac{\partial u(\mathbf{r}')}{\partial n'} - \frac{\partial u_2(\mathbf{r}';\mathbf{r})}{\partial n'}\right\}\\ & - \{u(\mathbf{r}') - u_2(\mathbf{r}';\mathbf{r})\}\frac{\partial^3 G(\mathbf{r}';\mathbf{r})}{\partial x_j \partial x_i \partial n'}\Bigg] d\Gamma' - \eta\frac{\partial^2 u(\mathbf{r})}{\partial x_j \partial x_i}.\end{aligned} \tag{29}$$

Since the function $u(\mathbf{r})$ satisfies the Laplace equation, (29) is reduced to simultaneous equations of five unknowns by replacing $\partial^2 u(\mathbf{r})/\partial x_2^2$ for $-\partial^2 u(\mathbf{r})/\partial x_1^2$.

In Ref. [6], a system of coupled equations, which is similar to but different from (27) and (29) and involves an arbitrary constant for the second-order derivatives, was derived on the basis of the evaluation of the numerical errors in boundary integrals.

The method described in this section, which is termed Method B in the remainder of this paper, has a favorable feature that it does not need to find boundary reference points, in contrast to Method A. On the other hand, the regularization in Method B employs the differences such as $u(\mathbf{r}') - u_k(\mathbf{r}';\mathbf{r})$, which are not expected to become zero, but small, in general, for the calculation of points close to boundaries, contrary to those in Method A. Finally, we point out here that the formulas in Methods A and B are proved to be equivalent to those in the conventional BEM, (1)–(3), unless there are numerical errors in the computation of the line integrals.

3. NUMERICAL RESULTS

Potential derivatives are here calculated by the conventional formulas (2) and (3), and the methods presented in this paper, for two simple potential problems. In all the computations, the Gaussian quadrature with 12 integration points is employed for the line integrals on each boundary element.

Figure 2 shows the first problem. The potential in the square is exactly given by $u = a(1 - x_1/a)$. The boundary of the square is uniformly subdivided into 12 line elements, in which the potential u and its normal derivative $\partial u/\partial n$ are interpolated by linear and constant functions, respectively. The boundary values u and $\partial u/\partial n$ are set to the exact ones on those elements. In this case, the above interpolation suffices to exactly express the boundary values. Calculation points are taken to be $(l_e d - a, 0)$, where d denotes the distance normalized with respect to element length $l_e = 2a/3$. Of course, this simple problem does not require any numerical integrations for boundary integrals. We dare to employ the Gaussian quadrature, however, for boundary integrals to see the basic performance of the present methods.

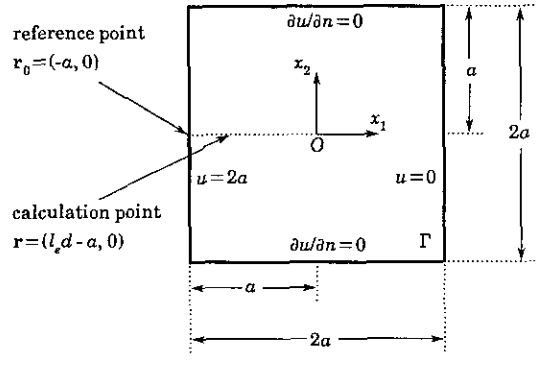

FIG. 2. Potential field in a square. The boundary of the square is uniformly subdivided into 12 line elements of length l_e, in which the potential and its normal derivative are interpolated by linear and constant functions, respectively.

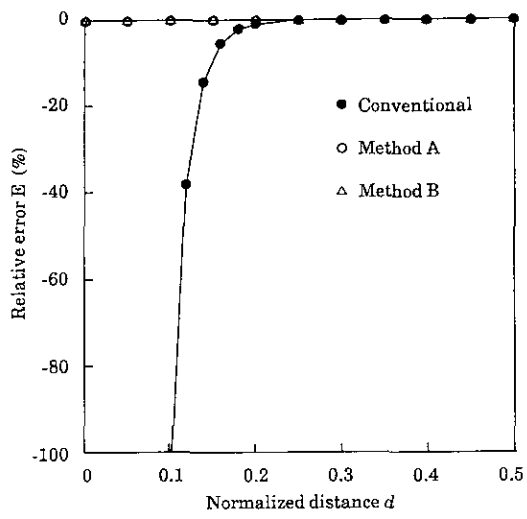

FIG. 3. Error distribution of $\partial u/\partial x_1$ near boundary for the problem shown in Fig. 2.

The errors in $\partial u/\partial x_1$ near a boundary are shown in Fig. 3. The relative error E is here defined by $(\partial u/\partial x_1 - u_{x1})/u_{x1} \times 100$ (%), where u_{x1} is the exact value of the potential derivative in x_1-direction. It is seen in Fig. 3 that both methods presented in this paper give accurate results even in the vicinity of the boundary, while the conventional formula yields tremendous errors there.

The second test problem is shown in Fig. 4, in which a cylindrical magnetic material of radius a is immersed in a uniform, static magnetic field $\mathbf{H}_0 = (H_0, 0)$. In this problem, the potential u, which is here defined by $\mathbf{H} = -\nabla u$, is analytically expressed in the form $H_0 x_1[a^2(\mu_r - 1)/(\mu_r + 1) - r^2]/r^2$ for $r \geq a$ and $-2H_0\mu_r x_1/(1 + \mu_r)$ for $r \leq a$, where μ_r denotes the relative permeability of the magnetic material and $r = \sqrt{x_1^2 + x_2^2}$. The boundary of the cylinder is uniformly subdivided into 36 quadratic elements of length $l_e = 2a\pi/36$. Each element has three nodal points, at which the exact values of the potential and normal derivative are provided. Calculation points are located at $(a + l_e d, 0)$, where d represents again the normalized distance. Note here that, in this problem, the normal derivatives of potentials on Γ are discontinuous. Thus the different analytical values of the normal derivatives are used in boundary integrals for inner and outer problems.

Figures 5a and 5b show the resultant error distributions for the first-order derivative $\partial u/\partial x_1$. The relative error E is defined by $(\partial u/\partial x_1 - u_{x1})/u_{x1}(0) \times 100$ (%), where $u_{x1}(0)$ is the exact value of $\partial u/\partial x_1$ at $d = +0$. Actually, in this problem as well as in the first one, the conventional method yields considerable errors in a zone adjacent to the boundary. Contrary to this, both methods proposed in this paper provide accurate results at all the points.

The results for the second derivative $\partial^2 u/\partial x_1^2$ are shown in Figs. 6a and 6b. In this case, the relative error E is defined by $(\partial^2 u/\partial x_1^2 - u_{x1x1})/u_{x1x1}(0) \times 100$ (%), where $u_{x1x1}(0)$ is the exact value of $\partial^2 u/\partial x_1^2$ at $d = +0$. In this case, the conventional method fails at points farther from the boundary due to the strong singularity in the kernel, on comparison with Figs. 5a and 5b. While the extreme approach of the calculation points to the boundary slightly deteriorates the accuracy of Method B, Method A computes $\partial^2 u/\partial x_1^2$ in good accuracy at all the calculation points.

4. DISCUSSION

In the previous section, the numerical examples show that the present regularization methods clearly improve the accuracy in potential derivatives. We here evaluate the numerical errors produced by the methods to see those performances from a general point of view.

We begin with the error analysis for Method A. The coordinates x_i, potential u, and its spatial derivatives are assumed to be normalized by the appropriate corresponding values hereafter. The difference $\Delta u(\mathbf{r})^{CV}$ between the potential value numerically obtained by (1) and the exact value is decomposed into the error ε_l from numerical integration and the other error ε from, e.g., rounding, truncation, and discretization of a system, as

$$\Delta u(\mathbf{r})^{CV} = \varepsilon + \varepsilon_l. \tag{30}$$

The present method cannot control the error ε but it reduces the integration error ε_l as shown below.

The substitution of the Taylor expansion of the potential $u(\mathbf{r}')$ in (1) around a boundary reference point $\mathbf{r}_0$ into (1) gives

$$\Delta u(\mathbf{r})^{CV} = \varepsilon + \delta(\mathbf{r})u(\mathbf{r}_0) + \delta_j(\mathbf{r}_0, \mathbf{r})\frac{\partial u(\mathbf{r}_0)}{\partial x_j} + \delta_{jk}(\mathbf{r}_0, \mathbf{r})\frac{\partial^2 u(\mathbf{r}_0)}{\partial x_j \partial x_k} + \cdots, \tag{31}$$

where

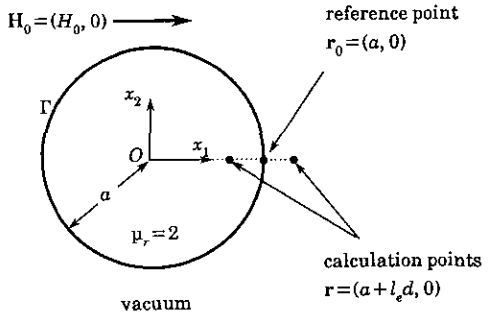

FIG. 4. A cylindrical magnetic material immersed in a uniform magnetic field. The boundary of the cylinder is uniformly subdivided into 36 curved elements of length l_e, in which the potential and its normal derivative are approximated by the quadratic interpolation functions.

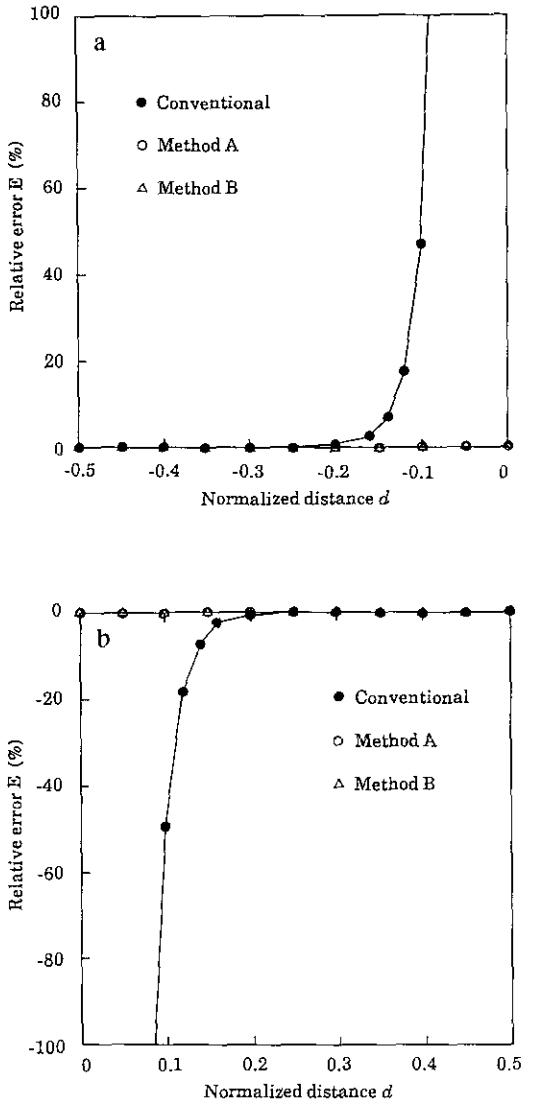

FIG. 5. Error distribution of $\partial u/\partial x_1$: (a) inside the cylinder shown in Fig. 4; (b) outside the cylinder shown in Fig. 4.

$$\delta(\mathbf{r}) = -\Delta \int_\Gamma \frac{\partial G(\mathbf{r}';\mathbf{r})}{\partial n'} d\Gamma', \tag{32a}$$

$$\delta_j(\mathbf{r}_0,\mathbf{r}) = \Delta \int_\Gamma \left[n_j' G(\mathbf{r}';\mathbf{r}) - (x_j' - x_{j0}) \frac{\partial G(\mathbf{r}';\mathbf{r})}{\partial n'} \right] d\Gamma', \tag{32b}$$

$$\delta_{jk}(\mathbf{r}_0,\mathbf{r}) = \Delta \int_\Gamma \left[n_j'(x_k' - x_{k0}) G(\mathbf{r}';\mathbf{r}) - \frac{1}{2}(x_j' - x_{j0})(x_k' - x_{k0}) \Big/ \frac{\partial G(\mathbf{r}';\mathbf{r})}{\partial n'} \right] d\Gamma', \tag{32c}$$

and

$$\Delta \int_\Gamma \cdot d\Gamma = \left[\int_\Gamma \cdot d\Gamma \right]_{\text{numerical}} - \left[\int_\Gamma \cdot d\Gamma \right]_{\text{exact}}. \tag{33}$$

One observes that the quasi singularities in boundary integrals in (32a) and (32b) are weaker than those in (32b) and (32c), respectively.

The application of the Taylor expansion to (12) also yields the error $\Delta u(\mathbf{r})^{\text{A}}$ for Method A as

$$\Delta u(\mathbf{r})^{\text{A}} = \varepsilon + \delta_j(\mathbf{r}_0,\mathbf{r}) \frac{\partial u(\mathbf{r}_0)}{\partial x_j} + \delta_{jk}(\mathbf{r}_0,\mathbf{r}) \frac{\partial^2 u(\mathbf{r}_0)}{\partial x_j \partial x_k} + \cdots. \tag{34}$$

By comparing (31) and (34), we see that Method A eliminates the error $\delta(\mathbf{r})u(\mathbf{r}_0)$ in (31). Similarly, one can estimate the errors $\Delta \partial u(\mathbf{r})/\partial x_j$ in the first derivatives of a potential, that is,

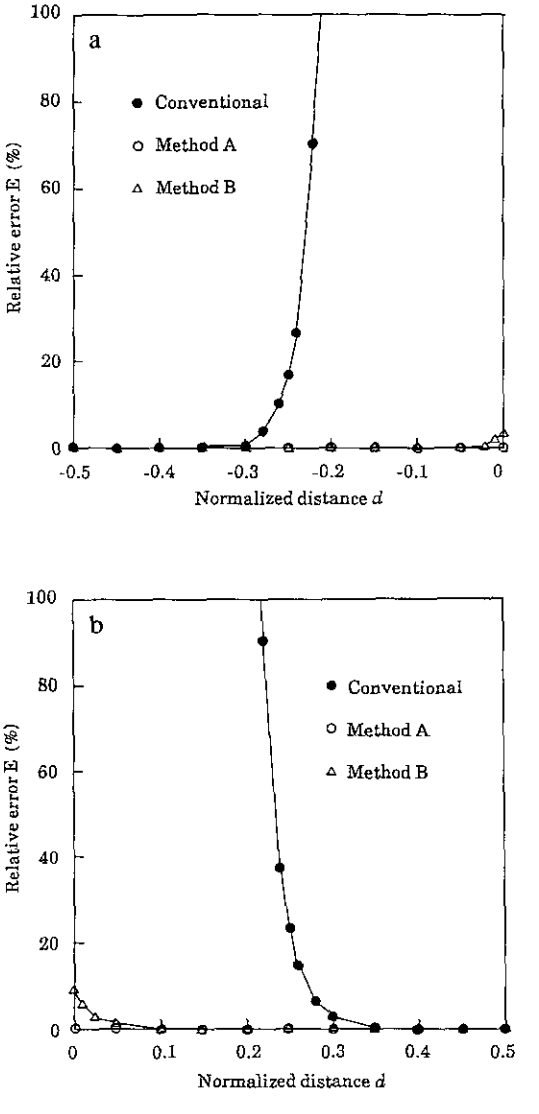

FIG. 6. Error distribution of $\partial^2 u/\partial x_1^2$: (a) inside the cylinder shown in Fig. 4; (b) outside the cylinder shown in Fig. 4.

$$\left[\Delta\frac{\partial u(\mathbf{r})}{\partial x_i}\right]^{CV} = \varepsilon^i + \delta^i(\mathbf{r})u(\mathbf{r}_0) + \delta^i_j(\mathbf{r}_0,\mathbf{r})\frac{\partial u(\mathbf{r}_0)}{\partial x_j} + \delta^i_{jk}(\mathbf{r}_0,\mathbf{r})\frac{\partial^2 u(\mathbf{r}_0)}{\partial x_j \partial x_k} + \cdots \tag{35}$$

for (2), and

$$\left[\Delta\frac{\partial u(\mathbf{r})}{\partial x_i}\right]^{A} = \varepsilon^i + \delta^i_{jk}(\mathbf{r}_0,\mathbf{r})\frac{\partial^2 u(\mathbf{r}_0)}{\partial x_j \partial x_k} + \cdots \tag{36}$$

for (18), where the superscript i above δ, δ_j, and δ_{jk} denotes the derivative with respect to x_i. In this case, the numerical errors relevant to δ^i and δ^i_j have been removed in (36).

When the boundary values $u(\mathbf{r}_0)$ and $\partial u(\mathbf{r}_0)/\partial x_j$ used in (12) and (18) have numerical errors $\Delta u(\mathbf{r}_0)$ and $\Delta[\partial u(\mathbf{r}_0)/\partial x_j]$, respectively, the errors (34) and (36) for Method A should be replaced by

$$\Delta u(\mathbf{r})^{A} = \varepsilon - \delta(\mathbf{r})\Delta u(\mathbf{r}_0) + \delta_j(\mathbf{r}_0,\mathbf{r})\frac{\partial u(\mathbf{r}_0)}{\partial x_j} + \delta_{jk}(\mathbf{r}_0,\mathbf{r})\frac{\partial^2 u(\mathbf{r}_0)}{\partial x_j \partial x_k} + \cdots, \tag{34'}$$

$$\left[\Delta\frac{\partial u(\mathbf{r})}{\partial x_i}\right]^{A} = \varepsilon^i - \delta^i(\mathbf{r})\Delta u(\mathbf{r}_0) - \delta^i_j(\mathbf{r}_0,\mathbf{r})\Delta\left[\frac{\partial u(\mathbf{r}_0)}{\partial x_j}\right] + \delta^i_{jk}(\mathbf{r}_0,\mathbf{r})\frac{\partial^2 u(\mathbf{r}_0)}{\partial x_j \partial x_k} + \cdots. \tag{36'}$$

As shown in (34') and (36'), Method A works well whenever the errors $\Delta u(\mathbf{r}_0)$ and $\Delta[\partial u(\mathbf{r}_0)/\partial x_j]$ are sufficiently small. Since Method A gave successful results to the test problems mentioned in the previous section, these conditions were thought to be well satisfied in those computations. One can evaluate the errors in the second derivatives of a potential in a similar way. Note here that the errors ε and ε^i remain unchanged in Method A.

We next perform an error analysis for Method B. By expanding the potential $u(\mathbf{r}')$ around a calculation point $\mathbf{r}$, we can rewrite the numerical error in (1) as

$$\Delta u(\mathbf{r})^{CV} = \varepsilon + \delta(\mathbf{r})u(\mathbf{r}) + \delta_j(\mathbf{r})\frac{\partial u(\mathbf{r})}{\partial x_j} + \delta_{jk}(\mathbf{r})\frac{\partial^2 u(\mathbf{r})}{\partial x_j \partial x_k} + \cdots. \tag{37}$$

Similarly, the error in (25) is evaluated as

$$\Delta u(\mathbf{r})^{B} = \frac{1}{1+\delta(\mathbf{r})}\left[\varepsilon + \delta_j(\mathbf{r})\frac{\partial u(\mathbf{r})}{\partial x_j} + \delta_{jk}(\mathbf{r})\frac{\partial^2 u(\mathbf{r})}{\partial x_j \partial x_k} + \cdots\right]. \tag{38}$$

As we see in (37) and (38), Method B eliminates the error $\delta(\mathbf{r})u(\mathbf{r})$ in (37). In addition, the residual error in the square bracket in (38) is divided by $1 + \delta(\mathbf{r})$. Since $\delta(\mathbf{r})$ usually takes the range $-0.5 \le \delta(\mathbf{r}) \le 0$ [3], this division seems to give undesirable effects for the accuracy in Method B. The term $\delta(\mathbf{r})u(\mathbf{r})$, which is usually dominant in (37), has vanished in (38) and thus this undesirable effect may seldom give rise to serious problems under wide conditions.

The error in the first derivatives of a potential is expressed as

$$\left[\Delta\frac{\partial u(\mathbf{r})}{\partial x_i}\right]^{CV} = \varepsilon^i + \delta^i(\mathbf{r})u(\mathbf{r}) + \delta^i_j(\mathbf{r})\frac{\partial u(\mathbf{r})}{\partial x_j} + \delta^i_{jk}(\mathbf{r})\frac{\partial^2 u(\mathbf{r})}{\partial x_j \partial x_k} + \cdots, \tag{39}$$

for the conventional formula (2), while the errors in (27) are given by

$$\begin{bmatrix} \Delta u(\mathbf{r})^{B'} \\ \left[\Delta\frac{\partial u(\mathbf{r})}{\partial x_1}\right]^{B} \\ \left[\Delta\frac{\partial u(\mathbf{r})}{\partial x_2}\right]^{B} \end{bmatrix} = \begin{bmatrix} 1+\delta(\mathbf{r}) & \delta_1(\mathbf{r}) & \delta_2(\mathbf{r}) \\ \delta^1(\mathbf{r}) & 1+\delta^1_1(\mathbf{r}) & \delta^1_2(\mathbf{r}) \\ \delta^2(\mathbf{r}) & \delta^2_1(\mathbf{r}) & 1+\delta^2_2(\mathbf{r}) \end{bmatrix}^{-1} \begin{bmatrix} \varepsilon + \delta_{jk}(\mathbf{r})\frac{\partial^2 u(\mathbf{r})}{\partial x_j \partial x_k} + \cdots \\ \varepsilon^1 + \delta^1_{jk}(\mathbf{r})\frac{\partial^2 u(\mathbf{r})}{\partial x_j \partial x_k} + \cdots \\ \varepsilon^2 + \delta^2_{jk}(\mathbf{r})\frac{\partial^2 u(\mathbf{r})}{\partial x_j \partial x_k} + \cdots \end{bmatrix}. \tag{40}$$

We see that (40) does not include the second and third terms in the right-hand side of (39), which may be much greater than the other terms. Instead, the residual errors in the column vector in right-hand side of (40), which are also expected to be small, are increased by $[A]^{-1}$ times, where $[A]$ represents the 3×3 matrix in (40). The accurate first derivatives obtained by Method B for the test problems suggest that this undesirable effect did not significantly deteriorate the accuracy in those computations.

The errors in (29) can be expressed in a matrix form similar to (40). The errors in the second derivatives in the vicinity of the boundary, shown in Figs. 6a and 6b, are probably ascribed to an error enhancement due to the matrix $[A]^{-1}$. In particular, the error observed in the outer region seems to have a great contribution from the non-zero third derivatives multiplied by $[A]^{-1}$, in contrast to that in the inner region in which the third derivatives equal zero everywhere.

The results obtained above will be further applied to the error analysis for the potential in the first test problem in the Appendix.

TABLE I

Errors in the Potential for the Problem Shown in Fig. 2

Normalized distance d	Conventional computed	Method A computed	Method B	
			Computed	Estimated
0.00	-5.047×10^{-1}	-4.627×10^{-3}	-9.253×10^{-3}	-9.255×10^{-3}
0.04	-1.201×10^{-1}	-8.530×10^{-4}	-2.774×10^{-3}	-2.774×10^{-3}
0.08	-1.836×10^{-2}	-1.251×10^{-4}	-6.227×10^{-4}	-6.227×10^{-4}
0.12	-2.621×10^{-3}	-1.805×10^{-5}	-1.225×10^{-4}	-1.225×10^{-4}
0.16	-3.823×10^{-4}	-2.694×10^{-6}	-2.296×10^{-5}	-2.295×10^{-5}
0.20	-5.823×10^{-5}	-4.217×10^{-7}	-4.277×10^{-6}	-4.276×10^{-6}

5. CONCLUSIONS

This paper has described two strategies for the accurate computation of the potential derivatives in BEM. Both methods presented in this paper are shown to effectively improve the accuracy in those computations. From the viewpoint of computer implementation, Method B is superior to Method A since the later involves a procedure for finding the boundary reference points. The error evaluation reveals that errors may be found in Method A, when there are significant errors in the values of potentials and its derivatives at the boundary reference point, and in Method B, when the residual errors including the value of higher order derivatives are not negligible. The present methods, however, reasonably regularize the quasi singularities in boundary integrals, except under extremely poor conditions.

Both methods are easily realized for computer applications without an involved algorithm. Moreover, the present formulas are readily derived by applying Green's theorem to the combinations of the algebraic function u_k and the fundamental solution. This suggests the possibility that those formulas can be extended to those for more practical field analyses in mechanics and electromagnetics.

APPENDIX

In general, it is difficult to directly compare the accuracy of Method A with that of Method B on the basis of the error evaluation given in the fourth section. Nevertheless, we can obtain a quantitative relation among numerical errors in some variables such as the potential for the first test problem shown in Fig. 2, to which we will apply the results obtained in the error analysis.

When the constant a and boundary reference point $\mathbf{r}_0$ are taken to be unity and $(-\frac{1}{2}, 0)$ in the first test problem, respectively, from (31), (34), (37), and (38), we estimate the numerical errors in $u(\mathbf{r})$ as

$$\Delta u(\mathbf{r})^{\mathrm{CV}} = \varepsilon + \delta(\mathbf{r}) - \delta_j(\mathbf{r}_0, \mathbf{r}), \tag{A1}$$

$$\Delta u(\mathbf{r})^{\mathrm{A}} = \varepsilon - \delta_j(\mathbf{r}_0, \mathbf{r}), \tag{A2}$$

$$\Delta u(\mathbf{r})^{\mathrm{CV}} = \varepsilon + \delta(\mathbf{r})u(\mathbf{r}) - \delta_j(\mathbf{r}), \tag{A3}$$

$$\Delta u(\mathbf{r})^{\mathrm{B}} = \frac{\varepsilon - \delta_j(\mathbf{r})}{1 + \delta(\mathbf{r})}. \tag{A4}$$

Note here that, in this problem, the spatial derivatives, except $\partial u/\partial x_1$, are equal to zero everywhere. In addition, the variables at $\mathbf{r}_0$ were assumed to have no errors in the above evaluation. From (A1) and (A2), we have

$$\delta(\mathbf{r}) = \Delta u(\mathbf{r})^{\mathrm{CV}} - \Delta u(\mathbf{r})^{\mathrm{A}}. \tag{A5}$$

The insertion of (A3) and (A5) into (A4) yields

$$\Delta u(\mathbf{r})^{\mathrm{B}} = \frac{\Delta u(\mathbf{r})^{\mathrm{CV}}[1 - u(\mathbf{r})] + \Delta u(\mathbf{r})^{\mathrm{A}} u(\mathbf{r})}{1 + \Delta u(\mathbf{r})^{\mathrm{CV}} - \Delta u(\mathbf{r})^{\mathrm{A}}}. \tag{A6}$$

This is the relation among the errors $\Delta u(\mathbf{r})^{\mathrm{CV}}$, $\Delta u(\mathbf{r})^{\mathrm{A}}$, and $\Delta u(\mathbf{r})^{\mathrm{B}}$. Table I displays those errors near $x_1 = -\frac{1}{2}$, with $\Delta u(\mathbf{r})^{\mathrm{B}}$ estimated from (A6). We see good agreement between the computed value of $\Delta u(\mathbf{r})^{\mathrm{B}}$ and the estimated one by (A6).

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